Partial Derivatives

(2.32) Definition. Let $z = f(x, y)$ be a function of two variables with Dom $f \subset R^2$. If x is changed to $x + \Delta x$ and y remains constant, then the change Δz in z is given by

$$\Delta z = f(x + \Delta x, y) - f(x, y)$$

If the incrementary ratio

$$\frac{\Delta z}{\Delta x} = \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

approaches a finite limit as $\Delta x \to 0$, then this limit is called the **partial derivative** of z (or of f) with respect to x and is denoted by

$$\frac{\partial z}{\partial x} \quad \text{or} \quad f_x \quad \text{or} \quad \frac{\partial f}{\partial x},$$ (to be read as partial f over partial x).

Similarly, the partial derivative of $z = f(x, y)$ with respect to y is

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

Other symbols for $\frac{\partial z}{\partial y}$ are f_y and $\frac{\partial f}{\partial y}$.

The calculation of partial derivatives of a given function $z = f(x, y)$ is quite easy. To obtain f_x, we find the derivative of f with respect to x holding y constant. Thus, if

$$z = x^2 \sin^2 y = f(x, y), \text{ then}$$

$$\frac{\partial z}{\partial x} = 2x \sin^2 y = f_x.$$

Similarly, $\frac{\partial z}{\partial y} = 2x^2 \sin y \cos y = x^2 \sin 2y = f_y$ holding x constant.

(2.33) Geometrical Meaning of Partial Derivatives.

Suppose $z = f(x, y)$ is a function of two variables. The graph of f is a surface. Let y be held constant by setting $y = b$. Then we are considering those points of the surface $z = f(x, y)$ for which $y = b$, a constant. But geometrically these points are the intersection of $z = f(x, y)$ and the plane $y = b$ which is a curve. On this curve z changes with x while y remains constant.

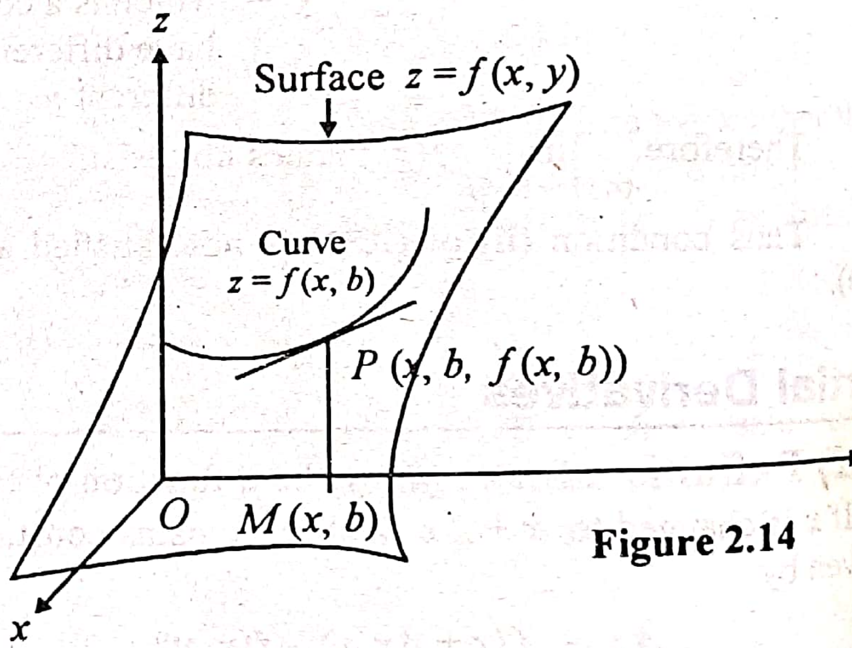

Figure 2.14

Therefore, $\frac{\partial z}{\partial x}$ is derivative of $z = f(x, b)$ with respect to x. Thus $\frac{\partial z}{\partial x}$ = slope of the tangent to this curve at P.

Similarly, $\frac{\partial z}{\partial y}$ is the gradient of the tangent at the point $(a, y, f(a, y))$ to the curve of intersection of $z = f(x, y)$ and the plane $x = a$, where a is constant.

Find the first order partial derivatives of the given functions **(Problems 15 – 22):**

15. $f(x, y) = x^{y^2}$

16. $f(x, y) = e^{x^2+y^2}$

17. $f(x, y) = \arctan\left(\frac{y}{x}\right)$

18. $f(x, y) = \arctan(x + y)$

19. $f(x, y) = e^{ax} \sin by$

20. $f(x, y) = \ln(x^2 + y^2)$

21. $f(x, y) = \ln\left[\frac{\sqrt{x^2 + y^2} - x}{\sqrt{x^2 + y^2} + x}\right]$

22. $f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$